Coherent state lattices and square integrability of representations

2003 J. Phys. A: Math. Gen. 36 11817

(http://iopscience.iop.org/0305-4470/36/47/009)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 11817–11835 PII: S0305-4470(03)68156-4

Coherent state lattices and square integrability of representations

Amétépé Lionel Hohouéto[1], **K Thirulogasanthar**[1], **S Twareque Ali**[1] **and Jean-Pierre Antoine**[2]

[1] Department of Mathematics and Statistics, Concordia University, Loyola Campus, 7141 Sherbrooke Street West, Montréal, Québec, H4B 1R6, Canada
[2] Institut de Physique Théorique, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium

E-mail: al_hohoueto@yahoo.fr, santhar@vax2.concordia.ca, stali@mathstat.concordia.ca and antoine@fyma.ucl.ac.be

Received 27 August 2003, in final form 10 October 2003
Published 12 November 2003
Online at stacks.iop.org/JPhysA/36/11817

Abstract

In this paper, we discretize the continuous theory of coherent states on a general semidirect product group $G = V \rtimes S$, where V is a vector space and $S \subset GL(V)$ is a semisimple connected Lie group. We show that it is always possible to construct a discrete frame associated with a unitary irreducible representation of G, which is square integrable modulo an affine admissible section. We also prove the converse result that the existence of a discrete frame associated with a unitary representation U of G, indexed by a homogeneous space Γ, implies the square integrability of U on Γ.

PACS numbers: 02.20.−a, 02.30.Nw

1. Introduction

Let G be a locally compact group and $U(g)$, $g \in G$, be a continuous unitary irreducible representation (UIR) of G on a (separable, complex) Hilbert space $\mathfrak{H}$. The representation $U(g)$ is said to be *square integrable* if there exists a nonzero vector $\eta \in \mathfrak{H}$ satisfying the *admissibility condition*

$$\int_G |\langle \eta_g \mid \eta \rangle|^2 \, \mathrm{d}\mu(g) < \infty. \qquad (1.1)$$

In that case, the following resolution of the identity holds:

$$\int_G |\eta_g\rangle\langle\eta_g| \, \mathrm{d}\mu(g) = I \qquad \eta_g = U(g)\eta \qquad (1.2)$$

where I is the identity operator on $\mathfrak{H}$ and μ is the left Haar measure of G. The vector η is called a (generalized) *analysing wavelet* and the η_g (generalized) *wavelets* or *coherent states*. For any vector $\phi \in \mathfrak{H}$, the function $F(g) = \langle \eta_g \mid \phi \rangle$ on G, is called the (generalized) *continuous wavelet transform* or *coherent state transform* of ϕ. If ϕ denotes a signal, then writing $F(g)$ in terms of the parameters of G enables one to analyse this signal in terms of these parameters. The resolution of the identity (1.2) also implies that, given a wavelet transform $F(g)$, the corresponding signal ϕ can be reconstructed uniquely. In fact, since the set of vectors η_g, $g \in G$, is overcomplete in $\mathfrak{H}$, it is possible to find a discrete set of points $g_i \in G$, $i = 1, 2, \ldots, N$, where N is generally infinite, such that the following *frame condition* holds:

$$\sum_{i=1}^{N} |\eta_{g_i}\rangle\langle\eta_{g_i}| = T \tag{1.3}$$

where T is a positive, bounded operator on $\mathfrak{H}$ with bounded inverse. In that case, knowledge of the values $F(g_i) = \langle \eta_{g_i} \mid \phi \rangle$ of the function F at the points g_i, $i = 1, 2, \ldots, N$, is enough to determine the signal ϕ uniquely. The set $\{F(g_i), i = 1, \ldots, N\}$ is called the *discretized wavelet transform* of ϕ and the vectors η_{g_i} are said to constitute a *discrete frame*.

It is hardly necessary to emphasize the usefulness of such a discretization. The continuous wavelet and coherent state transforms have rich theoretical structures and are extremely useful as tools for building signal transforms, adapted to various signal geometries. But for practical applications, it is the discretized versions of these transforms which are of greatest value and this provides the first motivation for the work presented in this paper. The second one comes from a related paper of Aniello *et al* [1], who have developed a scheme for the construction of discrete frames associated with groups of the form $G = \mathbb{R}^n \rtimes H$, with $H = \mathbb{R}_+ \times S$, and $n \in \mathbb{N}$, where S is a semisimple connected Lie group (these are the so-called wavelet-type groups). In their construction, the group G is assumed to have a square integrable unitary irreducible representation, induced from a unitary representation of the stabilizer of a fixed element k_0 of the dual $\widehat{\mathbb{R}}^n \simeq \mathbb{R}^n$ of $\mathbb{R}^n$. The element k_0 is assumed to have an orbit $\mathcal{O}^*$ of positive Lebesgue measure in $\mathbb{R}^n$. One then looks at the coherent state transform arising from this representation and discretizes it to obtain discrete frames. For the general sort of groups in which we are interested here, the corresponding induced representations are not square integrable with respect to the entire group. So we adopt the more general concept of square integrability modulo a subgroup and a section defined below (see [2] for an exhaustive discussion). Our main result is that the coherent state transform arising from such representations is again discretizable and one can again obtain discrete frames. A converse of this result also holds, namely, if one assumes the existence of a frame in this sense, then the corresponding representation is square integrable modulo a subgroup and a section. This generalizes a similar result in [3]. Analogous results were proved in an earlier paper [4] for the special case of the Poincaré group.

As in most treatments of this nature found in the literature, our analysis hinges on the existence of lattices in Lie groups. Given a locally compact topological group G and a closed subgroup X of G, X is said to be *uniform* (or co-compact) if the space G/X is compact. If X is discrete and G/X has an invariant finite measure, X is called a *lattice*. Then the key result is the following theorem, due to Borel [5].

Theorem 1.1 (Borel)**.** *Let G be a semisimple connected Lie group. Then, G contains both uniform and nonuniform lattices.*

In fact, it is always possible to generate a discrete frame for semisimple Lie groups with a square integrable UIR if a frame generator (in the sense of [6]) can be identified. The frame generator is a pair, consisting of a separated subset $\mathcal{X}$ of the group H and a compact subset $\mathcal{Q}$ of $\mathcal{O}^*$, such that $\mathcal{O}^*$ can be decomposed as the union of the translates of $\mathcal{Q}$ by elements of the separated set. Here a subset $\mathcal{X}$ of G is said to be *separated* if there exists a (compact) neighbourhood V of the identity in G such that

$$xV \cap yV = \emptyset \qquad \forall x, y \in \mathcal{X} \quad x \neq y.$$

$\mathcal{X}$ is also called a V-separated subset [7]. In the terminology of the latter reference, a family $\mathcal{X}_I = \{x_i : i \in I \subset \mathbb{N}\}$ of G is said to be *relatively separated* if it is the union of V-separated subsets. Moreover, we have the following characterization, on which the construction of discrete frames is based.

Lemma 1.2. *The following properties of $\mathcal{X}_I$ in G are equivalent:*

1. *The family $\mathcal{X}_I = \{x_i : i \in I \subset \mathbb{N}\}$ is relatively separated.*
2. *For all compact sets $K \subset G$, there exists a finite partition of the set of labels $I, I = \bigcup_{r=1}^{d} I_r$, such that each family $\{x_i K : i \in I_r\}$ consists of pairwise disjoint sets. Conversely, any relatively separated family can be obtained in this way.*
3. *Given a relatively compact set W with nonempty interior, we have*

$$\sup_{i\in I}\{\#\{j : x_j W \cap x_i W \neq \emptyset\}\} < \infty.$$

Actually, the ideas described above were already contained in the paper of Bohnké [8]. However, Aniello *et al* [1] were the first to provide a more systematic construction of this sort. Their construction is based on some simple topological results and indicates (with the help of the Borel theorem) a way to find the separated subset of the frame generator. Their method was successfully adapted to the Poincaré group $\mathcal{P}_+^{\uparrow}(1,3) = \mathbb{R}^4_{1,3} \rtimes SL(2,\mathbb{C})$ in [9]. This result suggests that it should be possible to extend the construction to an abstract semidirect product group, including groups of the same type which do not contain dilation subgroups. This is the case with most of the relativity groups used in the physical literature.

In this paper, we are also interested in the converse problem, which can be stated as follows: given a group G of the semidirect product type, and a (discrete) frame in the carrier space associated with a unitary irreducible representation U of G, does the existence of such a frame guarantee the square integrability of U? The answer to this question is *yes* when the frame is indexed by elements of G [3]. We show in this paper that it is still *yes* when the label space is a homogeneous space $\Gamma = G/H$ of the group.

The rest of this paper is organized as follows. In sections 2 and 3, respectively, we briefly review the continuous theory of coherent states on general semidirect product groups, and Aniello *et al*'s results on wavelet-type groups. In section 4, we first prove that the existence of a discrete frame, labelled by a homogeneous space and associated with a unitary representation of the group, implies the square integrability of the representation modulo a section. Then we present an explicit construction of discrete lattices of coherent states. As an example, we apply the method to the Euclidean group in two dimensions. We end the paper by some remarks.

2. Coherent states of general semidirect product groups revisited

Let V be a vector space and S be a subgroup of $GL(V)$, the linear group of V. Define an action of S on V by left translation, that is, a map $S \times V \to V$, $(s, x) \mapsto sx$, and form the group

$$G = V \rtimes S = \{g = (x, s) : x \in V, s \in S\} \tag{2.1}$$

with the multiplication law

$$(x, s)(x', s') = (x + sx', ss') \qquad (x, s)(x', s') \in G. \tag{2.2}$$

Most of the interesting groups encountered in the physics literature are of this form. Well-known examples are the connected affine group $\mathbb{R} \rtimes \mathbb{R}_+^*$ (also called the '$ax + b$' group), the similitude group $SIM(n) = \mathbb{R}^n \rtimes (\mathbb{R}_+^* \times SO(n))$, the Euclidean group in n dimensions $E(n) = \mathbb{R}^n \rtimes SO(n)$, the full Poincaré group $\mathcal{P}_+^\uparrow(1,3) = \mathbb{R}_{1,3}^4 \rtimes SL(2,\mathbb{C})$ and so on. In what follows, to be as general as possible, we will assume that the UIR used to build coherent states is not square integrable on the whole group, but modulo a subgroup H and a section [2]. The resulting coherent states are then labelled by points living in the quotient space G/H , which has the structure of a phase space.

2.1. A natural phase space

Let V^* denote the dual of V. The dual action of S on V^* is taken to be the map $S \times V^* \to V^*$, $(s, k) \mapsto s[k]$, defined by

$$\langle s[k], x\rangle = \langle k, s^{-1}x\rangle \qquad \forall x \in V \tag{2.3}$$

where $\langle .,.\rangle : V^* \times V \to \mathbb{R}$ is the duality pairing. Fix $k_0 \in V^*$. The orbit of k_0 under this action is denoted by $\mathcal{O}_{k_0} \equiv \mathcal{O}^*$ and its stabilizer by S_0. $\mathcal{O}^*$ is isomorphic to S/S_0 and, as a manifold, it is assumed to be of dimension m, in general lower than $n = \dim V^*$. For any $k \in \mathcal{O}^*$, let $T_k\mathcal{O}^* \equiv V_k^*$ and $T_k^*\mathcal{O}^* \equiv V_k$ denote the tangent and the cotangent spaces of $\mathcal{O}^*$ in k, respectively. Define the annihilator of V_k^* in V by

$$N_k = \{x \in V : \langle p, x\rangle = 0, \forall p \in V_k^*\} \tag{2.4}$$

and that of V_k in V^* by

$$N_k^* = \{p \in V^* : \langle p, x\rangle = 0, \forall x \in V_k\}. \tag{2.5}$$

It is well known that $V = V_k \oplus N_k$ and $V^* = V_k^* \oplus N_k^*$, for all $k \in \mathcal{O}^*$. Set $V_0 = V_{k_0}$, $N_0 = N_{k_0}$, $V_0^* = V_{k_0}^*$ and $N_0^* = N_{k_0}^*$. It can be easily proved [2] that the restriction of the dual action $S \times V^* \to V^*$ to $S_0 \times V^* \to V^*$ leaves $\mathcal{O}^*$ globally invariant, as does its push forward to V_0^*. By duality, N_0 is also left invariant. One can then form the subgroup $H_0 = N_0 \rtimes S_0$ of G and define the quotient space $\Gamma = G/H_0$. As a Borel space, Γ is isomorphic to $V_0 \times \mathcal{O}^*$. On the other hand, explicit computations [2] show that it is isomorphic, as a symplectic manifold, to the cotangent bundle $T^*\mathcal{O}^* = \bigcup_{k\in\mathcal{O}^*} V_k$ of $\mathcal{O}^*$, making it a natural phase space.

2.2. Coherent states and continuous frames

Here we briefly review the general construction of coherent states on semidirect product groups. Details can be found in [2].

In the general setting, the group has no square integrable UIR in the usual sense, but only a square integrable UIR modulo a closed subgroup H and an admissible section $\sigma : \Gamma = G/H \to G$, where Γ is assumed to carry an invariant measure μ. If H is the stabilizer of an element of V^*, the group admits a square integrable representation and the Gilmore–Perelomov method of construction of coherent states applies [10, 11].

Turning now to the problem of choosing a section, let $\sigma : \mathcal{O}^* \to S$ denote a global Borel section such that

$$\begin{aligned} &\sigma(k_0) = e \\ &\sigma(k)[k_0] = k \qquad \forall k \in \mathcal{O}^* \end{aligned} \tag{2.6}$$

and assume that there exists an open dense subset $\mathcal{O}$ of $\mathcal{O}^*$ on which σ is C^∞. This implies that

- for all s in S, there exists k in $\mathcal{O}^*$ and s_0 in S_0 such that $s = \sigma(k)s_0$,
- $\sigma(k)^{-1}v_k \in V_0$ for all k in $\mathcal{O}^*$ and $v_k \in V_k$.

From σ, a particular choice of section is given by the principal section $\sigma_{\text{pr}} : \Gamma \simeq V_0 \times \mathcal{O}^* \to G$,

$$\sigma_{\text{pr}}(q, p) = (\sigma(p)q, \sigma(p)) \qquad (q, p) \in \Gamma. \tag{2.7}$$

In view of the decomposition

$$(x, s) = (n_k + v_k, \sigma(k)s_0) = (v_k, \sigma(k))(\sigma(k)^{-1}n_k, s_0) \tag{2.8}$$

of any element of G into the product of a representative of its class modulo $H_0 = N_0 \rtimes S_0$ and an element of H_0, any other section $\widehat{\sigma} : \Gamma \to G$ is given by

$$\widehat{\sigma}(q, p) = \sigma_{\text{pr}}(q, p)(n(q, p), s_0(q, p)) \equiv (\widehat{q}, \Lambda(p)) \qquad (q, p) \in \Gamma \tag{2.9}$$

where $n : V_0 \times \mathcal{O}^* \to N_0$ and $s_0 : V_0 \times \mathcal{O}^* \to S_0$ are Borel functions. The function n is chosen affine in q, namely,

$$n(q, p) = \theta(p)q + \varphi(p) \tag{2.10}$$

where $\theta : \mathcal{O}^* \to \mathcal{L}(V)$ (the vector space of linear transformations of V) and $\varphi : \mathcal{O}^* \to N_0$ are C^∞ on $\mathcal{O}$ and $\operatorname{Ker}\theta = \operatorname{Ran}\theta = N_0$. The point s_0 is assumed to be a function of p only, that is,

$$s_0(q, p) = s_0(p) \qquad \forall q \in V_0 \tag{2.11}$$

and is also C^∞ on $\mathcal{O}$. These so-called *affine* sections must verify additional conditions before being suitable for the construction of coherent states. For specifying them, we need to define a local chart around k_0.

Let $\{e_\kappa, \kappa = 1, \ldots, n\}$ be a basis of V such that $\{e_\kappa, \kappa = 1, \ldots, m\}$ is a basis of V_0 and let $\{e^*_\kappa, \kappa = 1, \ldots, n\}$ be the dual basis in V^*, $\{e^*_\kappa, \kappa = 1, \ldots, m\}$ being a basis of V^*_0. It is always possible to find an open set $O_{k_0} \subset \mathcal{O}^*$, containing k_0, such that the map $\psi : O_{k_0} \to \mathbb{R}^m$ defined by

$$\psi(k)_\kappa \equiv k_\kappa = \langle k - k_0, e_\kappa \rangle \qquad \kappa = 1, \ldots, m \tag{2.12}$$

is a diffeomorphism. Using this local chart, any element k in O_{k_0} has the expansion

$$k = \sum_{\kappa=1}^{m} (k_\kappa + \alpha_\kappa) + n^*_k \tag{2.13}$$

with

$$\alpha_\kappa = \langle k_0, e_\kappa \rangle \qquad \kappa = 1, \ldots, m \qquad \text{and} \qquad n^*_k \in N^*_0$$

and is completely determined by the k_κ. Taking any $q = \sum_{\kappa=1}^{m} q^\kappa e_\kappa$ in V_0, we have

$$\langle k, q \rangle = \sum_{\kappa=1}^{m} (k_\kappa + \alpha_\kappa) q^\kappa. \tag{2.14}$$

With this system of coordinates, the affine section $\widehat{\sigma}$ is said to be *admissible* if

$$(a) \quad \forall p \in \mathcal{O}^* \qquad F(p)^*\big(O_{k_0}\big) \subset O_{k_0} \tag{2.15}$$

$$(b) \quad \mathcal{I}(p,k) \equiv \det J_{F(p)^*}\lceil O_{k_0} \neq 0 \qquad p \in \mathcal{O}^* \quad k \in O_{k_0} \tag{2.16}$$

where $F(p)^*$ is the adjoint of the map $F(p) \equiv \mathbb{I}_V + \theta(p)$ which appears in the explicit expression for $\widehat{\sigma}(q,p)$ and $J_{F(p)^*}\lceil O_{k_0}$ is the Jacobian matrix of $F(p)^*$ restricted to O_{k_0}. The class $\mathcal{S}_A$ of the affine admissible sections is invariant under the action of G, in the sense that, if $\widehat{\sigma}$ is an affine admissible section, then so is the section $\widehat{\sigma}'$ defined by

$$\widehat{\sigma}'(q,p) = g\sigma(g^{-1}\cdot(q,p)) \qquad \forall g \in G.$$

To determine the measure on Γ, we use the fact that the cotangent bundle $T^*\mathcal{O}^*$ of $\mathcal{O}^*$ comes equipped with a nondegenerate 2-form Ω, invariant under the group action. An invariant measure on $T^*\mathcal{O}^*$ can be built as follows. Consider a local chart (O,ψ) of $\mathcal{O}^*$, p being a point in $\mathcal{O}^*$ and $\{p_\kappa = \psi(p)_\kappa, \kappa = 1,\ldots,m\} \subset \mathbb{R}$ being its local coordinates. Let $\{\partial/\partial p_\kappa, \kappa = 1,\ldots,m\}$ be a basis of V_p^* and $\{\mathrm{d}p_\kappa, \kappa = 1,\ldots,m\}$ be the dual basis in V_p. Let $v_p = \sum_{\kappa=1}^m v_p^\kappa\,\mathrm{d}p_\kappa\ \left(v_p^\kappa \in \mathbb{R}, \forall\kappa\right)$ be an element of V_p. The 2-form Ω is explicitly

$$\Omega = \sum_{\kappa=1}^m \mathrm{d}v_p^\kappa \wedge \mathrm{d}p_\kappa$$

and the associated invariant measure is

$$\mathrm{d}\omega = \mathrm{d}v_p^1 \wedge\cdots\wedge \mathrm{d}v_p^m \wedge \mathrm{d}p_1 \wedge\cdots\wedge \mathrm{d}p_m = \mathrm{d}v_p \wedge \mathrm{d}p.$$

This measure in turn determines an invariant measure on Γ via the inverse of the Borel isomorphism $c : V_0 \times \mathcal{O}^* \to T^*\mathcal{O}^*$,

$$c(q,p) = (\sigma(p)q, p) \equiv (v_p, p) \qquad (v_p \in V_p).$$

To this end, let $\mathrm{d}q$ be the Lebesgue measure on V_0 and assume that $\mathcal{O}^*$ has an invariant measure ν under S. Then,

$$\mathrm{d}v_p = f(p)\,\mathrm{d}q \qquad f(p) = |\det[\sigma(p)^{-1}\lceil V_p]|^{-1} r(p)$$
$$\mathrm{d}\nu(p) = m(p)\,\mathrm{d}p$$

where r and m are smooth measurable functions on $\mathcal{O}^*$ which are positive and nonzero on O. The invariant measure ω of $T^*\mathcal{O}^*$ transforms locally on O under c into the measure on Γ,

$$\mathrm{d}\mu(q,p) = \frac{f(p)}{m(p)}\,\mathrm{d}q\,\mathrm{d}\nu(p).$$

Hence, the invariant measure on Γ is taken (globally) to be of the form

$$\mathrm{d}\mu(q,p) = \rho(p)\,\mathrm{d}q\,\mathrm{d}\nu(p)$$

where $\rho(p)$ is a measurable function that is positive and nonzero ν almost everywhere.

Finally, let U be the UIR of G induced from a unitary character χ of V and a UIR L of S_0 in a Hilbert space $\mathfrak{K}$ of dimension N. U is defined on the Hilbert space $\mathfrak{H} = \mathfrak{K} \otimes L^2(\mathcal{O}^*, \mathrm{d}\nu)$ by

$$(U(x,s)\phi)(k) = \mathrm{e}^{\mathrm{i}\langle k,x\rangle} L(h_0(s^{-1},k)^{-1})\phi(s^{-1}[k]) \qquad h_0(s,k) = \sigma(s[k])^{-1} s\sigma(k). \tag{2.17}$$

Consider next a set of vectors $\{\eta^j, j = 1,\ldots,N\} \subset \mathfrak{H}$, which are smooth functions on $\mathcal{O}^*$ having their supports contained in O_{k_0} and satisfying the following invariance condition under S_0:

$$U(0,s)\mathbb{F}U(0,s)^\dagger = \mathbb{F} \qquad \forall s \in S_0 \quad \text{where} \quad \mathbb{F} = \sum_{j=1}^N |\eta^j\rangle\langle\eta^j|. \tag{2.18}$$

Given an affine admissible section $\widehat{\sigma}$, we finally define the set of coherent states

$$\mathfrak{S}_{\widehat{\sigma}} = \left\{ \eta^j_{\widehat{\sigma}(q,p)} \equiv U(\widehat{\sigma}(q,p))\eta^j : j = 1, \ldots, N; (q,p) \in \Gamma \right\}.$$

One can verify [2] that the representation U is square integrable mod$(H_0, \widehat{\sigma})$ and that the operator

$$A_{\widehat{\sigma}} = \sum_{j=1}^{N} \int_{\Gamma} \mathrm{d}\mu(q,p) \big| \eta^j_{\widehat{\sigma}(q,p)} \big\rangle \big\langle \eta^j_{\widehat{\sigma}(q,p)} \big|$$

defines a rank-N (continuous) frame if and only if there exist two real numbers A and B such that

$$0 < A \leqslant (2\pi)^m \sum_{j=1}^{N} \int_{\mathcal{O}^*} \|\eta^j(\sigma(p)^{-1}[k])\|^2_{\mathfrak{K}} \frac{m(\sigma(p)^{-1}[k])}{|\mathcal{I}(p,k)|} \rho(p)\, \mathrm{d}\nu(p) \leqslant B < \infty.$$

3. Discrete frames, square integrability and wavelet groups

For the case of groups which are semidirect products of $\mathbb{R}^n$ and a unimodular group, say, $G = \mathbb{R}^n \rtimes H$, with $H = \mathbb{R}^*_+ \times S$, where S is a semisimple connected Lie group, Aniello *et al* [1, 3] have obtained two interesting results.

First, they have established that the existence of a discrete frame associated to a unitary representation U of G implies the square integrability of the representation, in the sense that the function $g \mapsto \langle \phi, U(g)\psi \rangle$ belongs to $L^2(G, \mu_G)$. Note that the representation U is *not* supposed to be irreducible. Their result, which applies typically to wavelet groups, can be summarized in the following proposition.

Proposition 3.1. *Let $\{x_l, l \in \mathbb{Z}^n\}$ be a maximal discrete subgroup of $\mathbb{R}^n$, generated by a basis $\{e_i, i = 1, \ldots, n\}$ of $\mathbb{R}^n$, $M(e_i)$ be the matrix having the e_i as its rows and $D = |\det M(e_i)|^{-1}$. Let $\{h_j, j \in J\}$ be an at most countable set in H, and ψ be a vector in $\mathfrak{H}$ such that $\{U(h_j x_l, h_j)\psi : l \in \mathbb{Z}^m, j \in J\}$ is a frame. Then,*

1. *If there is a compact set $K \subset H$ such that $\bigcup_{j\in J} h_j^{-1} K = H$, then U is a square integrable cyclic representation of G, and ψ is an admissible vector for U. Moreover, there exists $\beta > 0$ such that*
$$\int_G |\langle \phi, U(g)\psi \rangle|^2 \,\mathrm{d}\mu_G(g) \leqslant \beta D^{-1} \mu_H(K) \|\phi\|^2 \qquad \forall \phi \in \mathfrak{H}. \qquad (3.1)$$
2. *If C is a compact subset of H such that there exists a finite partition $\{J_1, \ldots, J_N\}$ of J satisfying, for any $p = 1, \ldots, N$, the condition*
$$i, j \in J_p \Rightarrow h_i^{-1} C \cap h_j^{-1} C = \emptyset$$
then, there exists $\alpha > 0$ such that
$$\alpha N^{-1} D^{-1} \mu_H(C) \|\phi\|^2 \leqslant \int_G |\langle \phi, U(g)\psi \rangle|^2 \,\mathrm{d}\mu_G(g) \qquad \forall \phi \in \mathfrak{H}. \quad (3.2)$$

Obviously, inequality (3.2) is interesting only if $\mu_H(C) > 0$, and U is square integrable. On the other hand, sufficient conditions for the existence of compact subsets of H satisfying the hypothesis of this proposition is assured by the Borel theorem and the following simple topological result [1].

Lemma 3.2

1. *There exists a discrete subgroup $\mathcal{X}_I = \{s_\tau : \tau \in I \subset \mathbb{N}\} \subset S$ and a compact $\mathcal{Q} \subset \mathcal{O}^*$ such that $\mathcal{O}^* = \bigcup_{\tau \in I} s_\tau[\mathcal{Q}]$.*

2. *For all compact $Q \subset \mathcal{O}^*$, there exists a finite partition $\{\mathcal{X}_r, r = 1, \ldots, d\}$ of $\mathcal{X}_I$ such that $s', s'' \in \mathcal{X}_r \Rightarrow s'[Q] \cap s''[Q] = \emptyset$.*
3. *For any open nonvoid subset Ω of $\mathcal{O}^*$, there exists a finite subset $\{s_\nu, \nu = 0, \ldots, N_0\} \subset S$ such that $\mathcal{O}^* = \bigcup_{\tau \in I} \bigcup_{\nu=0}^{N_0} s_\tau s_\nu[\Omega]$.*

The second result of [1], which also relies on the same lemma, reads as follows. Given a UIR U of G, induced from a UIR L of a compact subgroup $S_0 \subset S$, in the Hilbert space $\mathfrak{K}$, there exists in the representation space of U a discrete frame associated with it. The construction is based on the realization

$$(U(x,h)f)(k) = |\det h|^{1/2}\, \mathrm{e}^{\mathrm{i}\langle k,x\rangle} L(h,k) f(h^{-1}[k]) \qquad f \in \mathfrak{K} \otimes L^2(\mathcal{O}^*, \mathrm{d}\nu).$$

Here

$$L(h,k) = L(c(k)^{-1} h c(h^{-1}[k]))$$

where $c : \mathcal{O}^* \to H$ is a section for the action of H on $\mathcal{O}^*$, which is assumed to be locally continuous around k_0. Taking a cyclic vector v in $\mathfrak{K}$, there exists $\{s_i, i = 1, \ldots, N\} \subset S_0$ and a closed ball $\mathcal{B} \subset \mathcal{O}^*$ centred in k_0 such that

$$\alpha \|f\|_{\mathfrak{K}}^2 \leqslant \sum_{i=1}^{N} |\langle L(s_i,k)v, f\rangle_{\mathfrak{K}}|^2 \leqslant \beta \|f\|_{\mathfrak{K}}^2 \qquad \forall f \in \mathfrak{K} \quad \text{and} \quad k \in \mathcal{B}$$

where α, β are two positive numbers. Considering then the closed ball $\mathcal{B}$, the discrete subgroup $\mathcal{X}$ of lemma 3.2, an open subset $\mathcal{Y}$ of $\mathcal{O}^*$ such that $k_0 \in s_i[\mathcal{Y}] \subset \mathcal{B}$ for all $i = 1, \ldots, N$ and the set $X = \bigcap_{i=1}^{N} s_i[\mathcal{Y}]$, there exists a finite family $\{h_j, j = 1, \ldots, M\} \subset H$ such that

$$\mathcal{O}^* = \bigcup_{h \in \mathcal{X}} \bigcup_{j=1}^{M} h h_j [X].$$

Define finally, for $\psi \in \mathfrak{K} \otimes L^2(\mathcal{O}^*, \mathrm{d}\nu)$, the vector

$$\psi(k) = v\chi_y(k) \qquad k \in \mathcal{O}^*$$

where χ_y is the characteristic function of the set $\mathcal{Y}$. Then, for $h \in \mathcal{X}, 1 \leqslant j \leqslant M, 1 \leqslant i \leqslant N, x_l = 2\pi l \in \mathbb{R}^n, (l \in \mathbb{Z}^n)$, the countable set of vectors $\psi_{h,j,i,l} = U(hh_j s_i x_l, hh_j s_i)\psi$ constitutes a frame in $\mathfrak{K} \otimes L^2(\mathcal{O}^*, \mathrm{d}\nu)$.

4. Extension to an abstract semidirect product group

In this section we extend the results of Aniello *et al* to an abstract semidirect product group $G = V \rtimes S$ as defined in section 2. We first prove that the existence of a discrete frame indexed by a homogeneous space $\Gamma = G/H$ in the carrier space of a unitary representation U of G implies the square integrability of U on Γ (or modulo a section). Next, we show that it is always possible to build a discrete frame indexed by such a homogeneous space, using a UIR of G induced from a finite dimensional UIR of a closed subgroup.

4.1. Homogeneous spaces, frames and square integrability

Let $G, k_0, V_0, S_0, H_0, \Gamma, \mathcal{O}^*, \sigma, \sigma_{\mathrm{pr}}$ and $\widehat{\sigma}$ be as in section 2. Let U be a unitary representation of G in a Hilbert space $\mathfrak{H}$. Consider a lattice $\{q_l, l \in \mathbb{Z}^m\}$ in V_0, a discrete subset $\{p_n, n \in J \subset \mathbb{N}\}$ of $\mathcal{O}^*$ containing k_0, and assume there exists a set $\{\eta^j, j = 1, \ldots, N\}$ of linearly independent vectors in $\mathfrak{H}$ such that $\{\eta^j_{l,n} \equiv U(\sigma_{\mathrm{pr}}(-q_l, p_n))\eta^j : l \in \mathbb{Z}^m; n \in \mathbb{N};$

$j = 1, \ldots, N\}$ is a frame. Assume now the frame condition, that is, there exist two real numbers $0 < A \leqslant B < \infty$ such that, for all $\phi \in \mathfrak{H}$,

$$A\|\phi\|^2 \leqslant \sum_{j=1}^{N} \sum_{l\in\mathbb{Z}^m} \sum_{n\in J} \left|\left\langle \eta^j_{l,n}, \phi\right\rangle\right|^2 \leqslant B\|\phi\|^2. \tag{4.1}$$

Proposition 4.1. *For every ϕ in $\mathfrak{H}$, the function $(q, p) \mapsto |\langle U(\sigma_{\rm pr}(q, p))\eta^j, \phi\rangle|$ is in $L^2(\Gamma, \mu)$.*

Proof. Using the frame condition (4.1), we have that, for all (q, p) in Γ, and all ϕ in $\mathfrak{H}$,

$$A\|\phi\|^2 \leqslant \sum_{j,l,n} \left|\left\langle \eta^j_{l,n}, U(\sigma_{\rm pr}(q, p)^{-1})\phi\right\rangle\right|^2 \leqslant B\|\phi\|^2. \tag{4.2}$$

For any nonnegative function $g \in L^1(\Gamma, \mu)$, we can write

$$\begin{aligned} A\|g\|_1\|\phi\|^2 &\leqslant \int_\Gamma \mathrm{d}\mu(q, p) g(q, p) \sum_{j,l,n} |\langle \eta^j, U(\sigma_{\rm pr}(-q_l, p_n)^{-1}\sigma_{\rm pr}(q, p)^{-1})\phi\rangle|^2 \\ &\leqslant B\|g\|_1\|\phi\|^2. \end{aligned} \tag{4.3}$$

Applying the monotone convergence theorem to the integral, considering the term in k_0 (which corresponds to an index n_0) and using the group law, we obtain that

$$I_{n_0} = \sum_{j,l} \int_\Gamma \mathrm{d}\mu(q, p) g(q, p) |\langle \eta^j, U(-q + q_l, \sigma(p)^{-1})\phi\rangle|^2 \leqslant B\|g\|_1\|\phi\|^2. \tag{4.4}$$

The change of variables $q \mapsto q + q_l$, together with the invariance under translations of the (Lebesgue) measure $\mathrm{d}q$ on V_0, gives

$$\begin{aligned} I_{n_0} &= \sum_{j,l} \int_\Gamma \mathrm{d}\mu(q, p) g(q + q_l, p) |\langle \eta^j, U(-q, \sigma(p)^{-1})\phi\rangle|^2 \\ &= \sum_j \int_\Gamma \mathrm{d}\mu(q, p) \sum_l g(q + q_l, p) |\langle U(\sigma_{\rm pr}(q, p))\eta^j, \phi\rangle|^2. \end{aligned} \tag{4.5}$$

Consider now a function $\Theta \in \mathcal{C}_c(V_0)$, the space of compactly supported continuous functions on V_0, such that

$$\Theta \geqslant 0 \qquad \text{and} \qquad \int_{V_0} \mathrm{d}q\, \Theta(q) = 1.$$

Let K be compact subset of $\mathcal{O}^*$, and, for all α in $\mathbb{N}$, define the function $g_\alpha : \Gamma \to \mathbb{R}_+$ by

$$g_\alpha(q, p) = \frac{1}{\alpha^m} \Theta\left(\frac{1}{\alpha}\rho(p)q\right) \chi_K(p). \tag{4.6}$$

We have that $g_\alpha \in L^1(\Gamma, \mu)$, and $\|g_\alpha\| = \nu(K)$. Replacing g by g_α in (4.5), I_{n_0} becomes

$$I_{n_0} = I_{n_0,\alpha} = \sum_j \int_\Gamma \mathrm{d}\mu(q, p) S_\alpha(q, p) \chi_K(p) |\langle U(\sigma_{\rm pr}(q, p))\eta^j, \phi\rangle|^2 \tag{4.7}$$

with

$$S_\alpha(q, p) = \frac{1}{\alpha^m} \sum_{l\in\mathbb{Z}^m} \Theta\left(\frac{1}{\alpha}\rho(p)[q + q_l]\right). \tag{4.8}$$

Since Θ is compactly supported and continuous, the restriction of S_α to any compact subset of Γ is a finite sum of continuous terms, hence it is continuous. Moreover, we have

$$\lim_{\alpha\to\infty} S_\alpha(q, p) = D \int_{V_0} \mathrm{d}q\, \Theta(\rho(p)q) = D\rho(p)^{-1}. \tag{4.9}$$

Let $\{e_i, i = 1, \ldots, m\}$ be a basis of V_0 generating the lattice $\{q_l, l \in \mathbb{Z}^m\}$, and $M(e_i)$ be the matrix having the e_i as its rows, $D = |\det M(e_i)|^{-1}$. Application of Fatou's lemma to (4.7) now gives

$$\sum_j \int_\Gamma \mathrm{d}\mu(q,p) D\rho(p)^{-1} \chi_K(p) |\langle U(\sigma_{\mathrm{pr}}(q,p))\eta^j, \phi\rangle|^2 \leqslant B\nu(K)\|\phi\|^2.$$

Taking the infimum of the positive function ρ^{-1} over the compact set K in the l.h.s. of this inequality, we obtain

$$D \inf_K \rho(p)^{-1} \sum_j \int_\Gamma \mathrm{d}\mu(q,p) |\langle U(\sigma_{\mathrm{pr}}(q,p))\eta^j, \phi\rangle|^2 \leqslant B\nu(K)\|\phi\|^2$$

or

$$\sum_j \int_\Gamma \mathrm{d}\mu(q,p) |\langle U(\sigma_{\mathrm{pr}}(q,p))\eta^j, \phi\rangle|^2 \leqslant \frac{B\nu(K)}{D \inf_K \rho(p)^{-1}} \|\phi\|^2. \tag{4.10}$$

Hence the function $(q,p) \mapsto |\langle U(\sigma_{\mathrm{pr}}(q,p))\eta^j, \phi\rangle|$ is square integrable on $\Gamma = V_0 \times \mathcal{O}^*$.

□

A few remarks are in order here.

1. Statement 1 of proposition 3.1 is a particular case of ours. It corresponds to the case $H_0 = \{(0, e_S)\}$ and $\sigma_{\mathrm{pr}} = \mathrm{Id}_G$, the identity map $G \to G$.
2. The above proof did not require U to be irreducible.
3. Assuming irreducibility of U and putting $\eta^j_{\sigma_{\mathrm{pr}}(q,p)} = U(\sigma_{\mathrm{pr}}(q,p))\eta^j$, the result in proposition 4.1 can be restated by saying that the operator

$$\mathcal{A}_{\sigma_{\mathrm{pr}}} = \sum_{j=1}^N \int_\Gamma \mathrm{d}\mu(q,p) \big|\eta^j_{\sigma_{\mathrm{pr}}(q,p)}\big\rangle\big\langle\eta^j_{\sigma_{\mathrm{pr}}(q,p)}\big|$$

is bounded on $\mathfrak{H}$. We then have the following corollary which stems from lemma 10.3.2 and theorem 10.3.3 in [2].

Corollary 4.2. *Assume in addition that U is induced from a UIR L of S_0, and that the operator $\mathbb{F}$ satisfies the invariance property (2.18). Then, for all other sections σ' of the form (2.9), with*

$$n(q,p) = n(p) \qquad \forall (q,p) \in \Gamma \tag{4.11}$$

we have

$$\mathcal{A}_{\sigma'} = \mathcal{A}_{\sigma_{\mathrm{pr}}} = c(\sigma_{\mathrm{pr}}) I_{\mathfrak{H}} \tag{4.12}$$

where $c(\sigma_{\mathrm{pr}})$ is a positive nonzero constant, and $I_{\mathfrak{H}}$ is the identity operator in $\mathfrak{H}$. In this case, the existence of a discrete frame implies the existence of a tight continuous frame, both generated by $\mathbb{F}$, and indexed by Γ.

4.2. Discrete frames associated with an abstract admissible affine section

From here on, U is assumed to be induced from a UIR L of S_0, and square integrable modulo H_0 and $\widehat{\sigma}$. In this section, we show that there always exists a discrete frame associated with U.

4.2.1. A constructive method. Take a set $\{\eta^j : j = 1, \ldots, N\}$ of linearly independent vectors in $\mathfrak{H}$ with compact supports $K_j \subset O_{k_0}$ such that, for $j = 1, \ldots, N$,

- $s_0(p)[K_j] \subset O_{k_0}$ for all $p \in \mathcal{O}^*$,
- $\psi(F(p)^* s_0(p)[K_j]) = R_j$ is a regular hyperparallelepiped in $\mathbb{R}^m$, for all p,
- $k_0 \in \overset{\circ}{K}_j$.

The first condition on the K_j, even though it seems strong, is realistic because, in most of the cases encountered in the literature, the orbit $\mathcal{O}^*$ is open and $O_{k_0} = \mathcal{O}^*$. On the other hand, we will see that, in the particular case of the principal section $\sigma_{\rm pr}$, for instance, there is no need to impose it. A concrete example of such supports is provided by S_0-invariant sets.

Set $O = \bigcap_{j=1}^N \overset{\circ}{K}_j$ and $K = \bigcup_{j=1}^N K_j$. O is open and contains k_0 and K is compact. There exists $\{s_\nu : \nu = 0, \ldots, N_0\} \subset S$ such that

$$\mathcal{O}^* = \bigcup_{\tau\in I} \bigcup_{\nu=0}^{N_0} s_\tau s_\nu [O]. \tag{4.13}$$

On the other hand, consider the union of sets $\bigcup_{p\in\mathcal{O}^*} \Lambda(p)^{-1}[O]$. The collection is an open covering of $\mathcal{O}^*$, hence of each K_j. Hence, there exists, for each j, a finite subcovering $\Omega_j = \bigcup_{n\in\Xi_j\subset\mathbb{N}} \Lambda(p_n)^{-1}[O]$ of each K_j. The set

$$\Omega_K = \Omega_1 \cup \bigcup_{j=2}^N \big(\Omega_j \big\backslash \textstyle\bigcup_{\ell=1}^{j-1} \Omega_\ell\big) = \displaystyle\bigcup_{n\in\Theta\subset\mathbb{N}} \Lambda(p_n)^{-1}[O]$$

provides a finite subcovering of K. Obviously, for each j, card $\Xi_j \leqslant$ card Θ. Define next the compact sets

$$Q'_j = \bigcup_{n\in\Xi_j} \Lambda(p_n)[K_j] \qquad \text{and} \qquad Q' = \bigcup_{j=1}^N Q'_j. \tag{4.14}$$

Consider now an orthonormal basis $\{f^j, j = 1, \ldots, N\}$ of $\mathfrak{K}$ and assume that each η^j is of the form

$$\eta^j = f^j \otimes \eta\chi_{K_j} \tag{4.15}$$

where η is a continuous complex valued function of $L^2(\mathcal{O}^*, {\rm d}\nu)$. For a general admissible affine section, a straightforward computation using the local chart $\big(O_{k_0}, \psi\big)$ shows that, for $k \in O_{k_0}$,

$$\langle k, \widehat{q}\rangle = \langle k, \sigma(p)F(p)q + \sigma(p)\varphi(p)\rangle = \langle F(p)^*\sigma(p)^{-1}[k], q\rangle + \langle k, \sigma(p)\varphi(p)\rangle. \tag{4.16}$$

For such a section, set

$$q^j_{n,l} = \big(q^j_{n,l_\kappa}\big)_{\kappa=1}^m \quad \text{with} \quad q^j_{n,l_\kappa} = [\text{measure}(R_{j,\kappa})]^{-1} 2\pi l_\kappa \qquad l = (l_\kappa) \in \mathbb{Z}^m$$
$$\mu^j = \text{volume}(R_j)$$

and define

$$e^j_{n,l}(k) = \begin{cases} (\mu^j)^{-1/2}\, {\rm e}^{-{\rm i}\langle k, \widehat{q}^{\,j}_{n,l}\rangle} & \text{if} \quad k \in \Lambda(p_n)[K_j] \\ 0 & \text{otherwise.} \end{cases} \tag{4.17}$$

Note that, because of the specific choice of the K_j, if $k \in \Lambda(p_n)[K_j]$, then $\sigma(p_n)^{-1}[k] \in O_{k_0}$. This implication could take the form of an equivalence when one requires each K_j to be (globally) invariant under the action of S_0. For instance, this means working with S_0-invariant supports. Such a treatment in the specific case of the Poincaré group in 1+3 dimensions can be found in [4]. The proof is given in the appendix.

Lemma 4.3. *For n and j fixed and*

$$\rho_n(k) = \left|\frac{d\psi(\sigma(p_n)^{-1}[k])}{d\nu(k)}\right| |\mathcal{I}(p_n, \sigma(p_n)^{-1}[k])| \qquad \textit{for} \quad k \in \Lambda(p_n)[K_j]$$

the set $\{e^j_{n,l} : l \in \mathbb{Z}^m\}$ *is an orthonormal basis of* $L^2(\Lambda(p_n)[K_j], \rho_n(k)\,d\nu(k))$.

The main result of this part is the following theorem.

Theorem 4.4. *The set*

$$\{\eta^j_{\tau,\nu,n,l} \equiv U(s_\tau s_\nu \Lambda(p_n)\widehat{q}^{\,j}_{l,n}, s_\tau s_\nu \Lambda(p_n))\eta^j : \tau \in I; \nu = 0, \ldots, N_0; \\ n \in \Theta; l \in \mathbb{Z}^m; j = 1, \ldots, N\}$$

is a (discrete) frame in $\mathfrak{H}$.

Proof. We have to show that, for all ϕ in $\mathfrak{H}$, there exist two real numbers A and B such that

$$0 < A\|\phi\|^2_{\mathfrak{H}} \leqslant \mathcal{A} = \sum_{j=1}^{N}\sum_{\tau\in I}\sum_{\nu=0}^{N_0}\sum_{n\in\Theta}\sum_{l\in\mathbb{Z}^m} \left|\langle\eta^j_{\tau,\nu,n,l}, \phi\rangle_{\mathfrak{H}}\right|^2 \leqslant B\|\phi\|^2_{\mathfrak{H}}. \tag{4.18}$$

We first note that

$$\left(s_\tau s_\nu \Lambda(p_n)\widehat{q}^{\,j}_{l,n}, s_\tau s_\nu \Lambda(p_n)\right) = (0, s_\tau s_\nu)\left(\Lambda(p_n)\widehat{q}^{\,j}_{l,n}, \Lambda(p_n)\right) \tag{4.19}$$

$$\begin{aligned}
\langle\eta^j_{\tau,\nu,n,l}, \phi\rangle_{\mathfrak{H}} &= \langle U(\Lambda(p_n)\widehat{q}^{\,j}_{l,n}, \Lambda(p_n))\eta^j, U(0, s_\tau s_\nu)\phi\rangle_{\mathfrak{H}} \\
&= \int_{\mathcal{O}^*} d\nu(k)\, e^{-i\langle k, \Lambda(p_n)\widehat{q}^{\,j}_{l,n}\rangle} \eta^j(\Lambda(p_n)^{-1}[k])^\dagger \\
&\quad \times L(h_0(\Lambda(p_n)^{-1}, k)^{-1})^{-1}(U(0, s_\tau s_\nu)^{-1}\phi)(k) \\
&= \int_{\mathcal{O}^*} d\nu(k)\, e^{-i\langle \Lambda(p_n)^{-1}[k], \widehat{q}^{\,j}_{l,n}\rangle} \eta^j(\Lambda(p_n)^{-1}[k])^\dagger \\
&\quad \times L(h_0(\Lambda(p_n), \Lambda(p_n)^{-1}[k]))^{-1}(U(0, s_\tau s_\nu)^{-1}\phi)(k) \\
&= \int_{\mathcal{O}^*} d\nu(k)\, e^{-i\langle k, \widehat{q}^{\,j}_{l,n}\rangle} \eta^j(k)^\dagger L(h_0(\Lambda(p_n), k))^{-1}(U(0, s_\tau s_\nu)^{-1}\phi)(\Lambda(p_n)[k]) \\
&= (\mu^j)^{1/2} \int_{\mathcal{O}^*} d\nu(k) \overline{e}^j_{l,n}(k) \langle (U(0, \Lambda(p_n))\eta^j) \\
&\quad \times (\Lambda(p_n)[k]), (U(0, s_\tau s_\nu)^{-1}\phi)(\Lambda(p_n)[k])\rangle_{\mathfrak{K}}
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
\mathcal{A} &= \sum_{j,\tau,\nu,n,l} \left|\langle\eta^j_{\tau,\nu,n,l}, \phi\rangle_{\mathfrak{H}}\right|^2 \\
&= \sum_{j,\tau,\nu,n,l} \mu^j \Bigg| \int_{\mathcal{O}^*} d\nu(k)\rho_n(k)\overline{e}^j_{l,n}(k)\rho_n(k)^{-1} \\
&\quad \times \langle (U(0, \Lambda(p_n))\eta^j)(\Lambda(p_n)[k]), (U(0, s_\tau s_\nu)^{-1}\phi)(\Lambda(p_n)[k])\rangle_{\mathfrak{K}} \Bigg|^2 \\
&= \sum_{j,\tau,\nu,n} \mu^j \int_{\mathcal{O}^*} d\nu(k)\rho_n(k)^{-1} |\langle (U(0, \Lambda(p_n))\eta^j) \\
&\quad \times (\Lambda(p_n)[k]), (U(0, s_\tau s_\nu)^{-1}\phi)(\Lambda(p_n)[k])\rangle_{\mathfrak{K}}|^2 \quad \text{(by Plancherel's theorem)} \\
&= \sum_{j,\tau,\nu}\sum_{n\in\Theta} \mu^j \int_{\mathcal{O}^*} d\nu(k)\rho_n(k)^{-1} \\
&\quad \times |\langle (U(0, \Lambda(p_n))\eta^j)(\Lambda(p_n)[k]), (U(0, s_\tau s_\nu)^{-1}\phi)(\Lambda(p_n)[k])\rangle_{\mathfrak{K}}|^2.
\end{aligned} \tag{4.21}$$

From (4.21) we have

$$\mathcal{A} \geqslant \sum_{j,\tau,\nu} \mu^j \int_{K_j} \mathrm{d}\nu(k)\rho_{n_0}(k)^{-1}|\eta(k)|^2|\langle f^j, (U(0, s_\tau s_\nu)^{-1}\phi)(k)\rangle_{\mathfrak{K}}|^2$$

(because there exists $n_0 \in \Theta$ such that $p_{n_0} = k_0$; implying that $\Lambda(p_{n_0}) = e_S$ + the fact that $\sum_{n\in\Theta}\ldots \geqslant$ (the term in n_0))

$$\geqslant \sum_{j,\tau,\nu} \mu^j \inf_{K_j} \rho_{n_0}(k)^{-1} \inf_{K_j} |\eta(k)|^2 \int_O \mathrm{d}\nu(k)|\langle f^j, (U(0, s_\tau s_\nu)^{-1}\phi)(k)\rangle_{\mathfrak{K}}|^2$$

$$\geqslant \left(\min_{j=1}^{N} \mu^j\right) \inf_K \rho_{n_0}(k)^{-1} \inf_K |\eta(k)|^2 \sum_{\tau,\nu} \int_O \mathrm{d}\nu(k) \times \sum_{j=1}^{N} |\langle f^j, L(h_0((s_\tau s_\nu)^{-1}, k))\phi((s_\tau s_\nu)^{-1}[k])\rangle_{\mathfrak{K}}|^2$$

$$\geqslant \left(\min_{j=1}^{N} \mu^j\right) \inf_K \rho_{n_0}(k)^{-1} \inf_K |\eta(k)|^2 \sum_{\tau,\nu} \int_O \mathrm{d}\nu(k)\|\phi((s_\tau s_\nu)^{-1}[k])\|^2_{\mathfrak{K}}$$

$$\geqslant \left(\min_{j=1}^{N} \mu^j\right) \inf_K \rho_{n_0}(k)^{-1} \inf_K |\eta(k)|^2 \sum_{\tau,\nu} \int_{s_\tau s_\nu[O]} \mathrm{d}\nu(k)\|\phi(k)\|_{\mathfrak{K}}$$

($k \mapsto s_\tau s_\nu[k]$ + invariance of the measure)

$$\geqslant \left(\min_{j=1}^{N} \mu^j\right) \inf_K \rho_{n_0}(k)^{-1} \inf_K |\eta(k)|^2 \int_{\bigcup_{\tau\in I}\bigcup_{\nu=0}^{N_0} s_\tau s_\nu[O]=\mathcal{O}^*} \mathrm{d}\nu(k)\|\phi(k)\|^2_{\mathfrak{K}}$$

$$\left(\text{because} \sum_{\tau,\nu} \int_{s_\tau s_\nu[O]} \ldots \geqslant \int_{\bigcup_{\tau\in I}\bigcup_{\nu=0}^{N_0} s_\tau s_\nu[O]} \ldots\right)$$

$$\geqslant \left(\min_{j=1}^{N} \mu^j\right) \inf_K \rho_{n_0}(k)^{-1} \inf_K |\eta(k)|^2 \|\phi\|^2_{\mathfrak{H}}$$

and

$$\mathcal{A} \leqslant \sum_{j,\tau,\nu,n} \mu^j \int_{K_j} \mathrm{d}\nu(k)\rho_n(k)^{-1}\|(U(0, \Lambda(p_n))\eta^j)(\Lambda(p_n)[k])\|^2_{\mathfrak{K}} \times \|(U(0, s_\tau s_\nu)^{-1}\phi)(\Lambda(p_n)[k])\|^2_{\mathfrak{K}} \quad \text{(by the Schwarz inequality)}$$

$$\leqslant \sum_{j,\tau,\nu,n} \mu^j \sup_{K_j} \rho_n(k)^{-1} \int_{\Lambda(p_n)[K_j]} \mathrm{d}\nu(k)|\eta(\Lambda(p_n)^{-1}[k])|^2\|(U(0, s_\tau s_\nu)^{-1}\phi)(k)\|^2_{\mathfrak{K}}$$

($k \mapsto \Lambda(p_n)^{-1}[k]$ + invariance of the measure + unitarity of U)

$$\leqslant \sum_{j=1}^{N} \mu^j \left(\sum_{n\in\Theta} \sup_{K_j} \rho_n(k)^{-1}\right) \sup_{K_j} |\eta(k)|^2 \sum_{\tau,\nu} \int_{Q'_j} \mathrm{d}\nu(k)\|(U(0, s_\tau s_\nu)^{-1}\phi)(k)\|^2_{\mathfrak{K}}$$

$$\leqslant N \left(\max_{j=1}^{N} \mu^j\right) \left(\sum_{n\in\Theta} \sup_K \rho_n(k)^{-1}\right) \sup_K |\eta(k)|^2 \sum_{\tau,\nu} \int_{Q'} \mathrm{d}\nu(k)\|\phi(s_\tau s_\nu[k])\|^2_{\mathfrak{K}}$$

(because $K_j \subset K$ and $Q'_j \subset Q'$)

$$\leqslant N \left(\max_{j=1}^{N} \mu^j\right) \left(\sum_{n\in\Theta} \sup_K \rho_n(k)^{-1}\right) \sup_K |\eta(k)|^2 \sum_{\tau,\nu} \int_{s_\tau s_\nu[Q']} \mathrm{d}\nu(k)\|\phi(k)\|^2_{\mathfrak{K}}$$

($k \mapsto (s_\tau s_\nu)^{-1}[k]$ + invariance of the measure)

$$\leqslant N\left(\max_{j=1}^{N}\mu^{j}\right)\left(\sum_{n\in\Theta}\sup_{K}\rho_n(k)^{-1}\right)\sup_{K}|\eta(k)|^2\sum_{\tau,\nu}\int_{s_\tau s_\nu[Q']}\mathrm{d}\nu(k)\|\phi(k)\|^2_{\mathfrak{K}}$$

$$\leqslant N\left(\max_{j=1}^{N}\mu^{j}\right)\left(\sum_{n\in\Theta}\sup_{K}\rho_n(k)^{-1}\right)\sup_{K}|\eta(k)|^2\sum_{\tau\in I}\sum_{\nu=0}^{N_0}\int_{s_\tau[\bigcup_{\nu=0}^{N_0}s_\nu[Q']]}\mathrm{d}\nu(k)\|\phi(k)\|^2_{\mathfrak{K}}$$

$\big($because $s_\nu[Q']\subset\bigcup_{\nu=0}^{N_0}s_\nu[Q']\big)$

$$\leqslant (N_0+1)N\left(\max_{j=1}^{N}\mu^{j}\right)\left(\sum_{n\in\Theta}\sup_{K}\rho_n(k)^{-1}\right)\sup_{K}|\eta(k)|^2$$

$$\times\sum_{r=1}^{d}\sum_{\tau\in I_r}\int_{s_\tau[\bigcup_{\nu=0}^{N_0}s_\nu[Q']]}\mathrm{d}\nu(k)\|\phi(k)\|^2_{\mathfrak{K}}$$

$\big($since $\bigcup_{\nu=0}^{N_0}s_\nu[Q']$ is compact, it follows from part 2 of lemma 3.2 that there exists a finite partition $(\mathcal{X}_{I_r}, r=1,\ldots,d)$ of $\mathcal{X}_I$ such that $s_\tau\big[\bigcup_{\nu=0}^{N_0}s_\nu[Q']\big]\bigcap s_{\tau'}\big[\bigcup_{\nu=0}^{N_0}s_\nu[Q']\big]=\emptyset, \forall\tau\neq\tau'\in I_r\big)$

$$\leqslant (N_0+1)N\left(\max_{j=1}^{N}\mu^{j}\right)\left(\sum_{n\in\Theta}\sup_{K}\rho_n(k)^{-1}\right)\sup_{K}|\eta(k)|^2$$

$$\times\sum_{r=1}^{d}\int_{\bigcup_{\tau\in I_r}s_\tau[\bigcup_{\nu=0}^{N_0}s_\nu[Q']]}\mathrm{d}\nu(k)\|\phi(k)\|^2_{\mathfrak{K}}$$

$$\leqslant (N_0+1)N\left(\max_{j=1}^{N}\mu^{j}\right)\left(\sum_{n\in\Theta}\sup_{K}\rho_n(k)^{-1}\right)\sup_{K}|\eta(k)|^2\sum_{r=1}^{d}\int_{\mathcal{O}^*}\mathrm{d}\nu(k)\|\phi(k)\|^2_{\mathfrak{K}}$$

$\big($because $\bigcup_{\tau\in I_r}s_\tau\big[\bigcup_{\nu=0}^{N_0}s_\nu[Q']\big]\subset\mathcal{O}^*\big)$

$$\leqslant d(N_0+1)N\left(\max_{j=1}^{N}\mu^{j}\right)\left(\sum_{n\in\Theta}\sup_{K}\rho_n(k)^{-1}\right)\sup_{K}|\eta(k)|^2\|\phi\|^2_{\mathfrak{H}}.$$

The inequalities (4.18) have thus been demonstrated with the lower and upper bounds

$$\begin{aligned} A &= \left(\min_{j=1}^{N}\mu^{j}\right)\inf_{K}\rho_{n_0}(k)^{-1}\inf_{K}|\eta(k)|^2 \\ B &= d(N_0+1)N\left(\max_{j=1}^{N}\mu^{j}\right)\left(\sum_{n\in\Theta}\sup_{K}\rho_n(k)^{-1}\right)\sup_{K}|\eta(k)|^2. \end{aligned} \tag{4.22}$$

□

In the particular case of the principal section σ_{pr}, the construction is the same but we proceed under simpler considerations. Since $s_0(p)=e_S$ and $\theta(p)=\tilde{0}$, we have $\Lambda(p_n)=\sigma(p_n)$ and $\rho_n(k)\equiv\rho(k)=|D_k\psi|$. The bounds of the frame obtained are then

$$\begin{aligned} A &= \left(\min_{j=1}^{N}\mu^{j}\right)\inf_{K}\rho(k)^{-1}\inf_{K}|\eta(k)|^2 \\ B &= d(N_0+1)N\left(\max_{j=1}^{N}\mu^{j}\right)\ \mathrm{Card}\,\Theta\sup_{K}\rho(k)^{-1}\sup_{K}|\eta(k)|^2. \end{aligned}$$

4.2.2. Application to the Euclidean group in two dimensions. We will now apply this construction to the two-dimensional Euclidean group, following [2] and [12]. The group is $E(2)=\mathbb{R}^2\rtimes SO(2)$ and its elements are of the form (x,α), identified with (x,r_α), where

$x \in \mathbb{R}^2$ and r_α is the usual 2×2 rotation matrix of angle $\alpha \in [0, 2\pi]$. The orbits of points in $\mathbb{R}^2$ are circles. We fix the element $k_0 = (1, 0)$ and take $\mathcal{O}^*$ as the unit circle S^1 while $S_0 = \{\mathbb{I}_2\}$. We use the canonical parametrization $\psi(p) = (\cos p, \sin p)$, $p \in (-\pi, \pi)$, of the unit circle and the coordinatization

$$(q, p) \in \Gamma \simeq \mathbb{R} \times S^1 \simeq T^*S^1 = \{(x, p) \in \mathbb{R} \times S^1 : x_1 p_1 + x_2 p_2 = 0\}.$$

The section is given by $\sigma(q, p) = ((-q \sin p, q \cos p), r_p)$. The invariant measure on Γ is $\mathrm{d}q\,\mathrm{d}p$ and the representation of $E(2)$ is

$$(U(x, \alpha)\phi)(\theta) = \mathrm{e}^{\mathrm{i}\langle x, \psi(\theta)\rangle}\phi(\theta - \alpha) \qquad \phi \in L^2(S^1, \mathrm{d}k).$$

A function $\eta \in L^2(S^1, \mathrm{d}k)$ is admissible if and only if the following three conditions are fulfilled,

$$\begin{cases} (1) \quad \operatorname{supp} \eta \subset (-\pi/2, \pi/2), \\ (2) \quad \eta \text{ is even}, \eta(-\theta) = \eta(\theta), \\ (3) \quad \int_{-\pi/2}^{\pi/2} \frac{|\eta(\theta)|^2}{\cos\theta}\,\mathrm{d}\theta < \infty. \end{cases}$$

The family of coherent states $\eta_{q,p} = U(\sigma(q, p))\eta$ leads to a resolution of the identity.

For this group, the construction described can be performed when one considers the uniform lattices

$$\mathcal{X}_N = \{s_\tau = r_{2\pi\tau/N} : \tau = 0, 1, \ldots, N - 1\}, \qquad N \text{ arbitrary in } \mathbb{N}$$

of $SO(2)$. Then, using the open interval $J = (-\pi/2, \pi/2)$, we can write

$$S^1 = \bigcup_{\tau=0}^{N-1} \bigcup_{\nu\in\{0,1\}} s_\tau s_\nu[J] \tag{4.23}$$

where $s_\nu = r_{\nu\pi/N}$, $\nu = 0, 1$. A finite covering of $[-\pi/2, \pi/2]$ (hence of supp η) is obtained from the set $\{\sigma(p)^{-1}[J] : p \in S^1\}$ when we take $(p_n)_{n=-1,0,1} = (n\pi/N)_{n=-1,0,1}$. Finally, a frame is obtained when we consider the family

$$\mathcal{F} = \{\eta_{\tau,\nu,n,l} = U(s_\tau s_\nu \sigma(p_n)\widehat{q}_{n,l}, s_\tau s_\nu \sigma(p_n))\eta : \tau = 0, \ldots, N - 1; \\ \nu = 0, 1; n = -1, 0, 1; l \in \mathbb{Z}\}$$

where

$$\begin{cases} \widehat{q}_{n,l} = (-q_l \sin p_n, q_l \cos p_n) \\ q_l = 2\pi l \quad l \in \mathbb{Z} \end{cases} \qquad \text{and} \qquad \sigma(p_n) = r_{p_n}.$$

Let us remark that $s_\tau s_\nu \sigma(p_n) = r_{\theta_{\tau,\nu,n}}$, with

$$\theta_{\tau,\nu,n} = \frac{2\tau\pi}{N} + \frac{\nu\pi}{N} + \frac{n\pi}{N} = \frac{(2\tau + \nu + n)\pi}{N} \qquad \tau = 0, \ldots, N - 1 \quad \nu = 0, 1 \quad n = -1, 0, 1$$

and

$$s_\tau s_\nu \sigma(p_n)\widehat{q}_{n,l} = -q_l \begin{pmatrix} \sin(\pi - \theta_{\tau,\nu,n} - p_n) \\ \cos(\pi - \theta_{\tau,\nu,n} - p_n) \end{pmatrix}.$$

Then the frame vectors are of the form

$$\eta_{\tau,\nu,n,l}(\theta) = \mathrm{e}^{\mathrm{i}q_l \sin(\theta - \theta_{\tau,\nu,n} - p_n)}\eta(\theta - \theta_{\tau,\nu,n}) \\ = \mathrm{e}^{2\mathrm{i}\pi l \sin(\theta - (2\tau + \nu + 2n)\pi/N)}\eta\left(\theta - \frac{(2\tau + \nu + n)\pi}{N}\right)$$

or, more concisely,

$$\eta_{n,m,l}(\theta) = \mathrm{e}^{2\mathrm{i}\pi l \sin(\theta - (n - m)\pi/N)}\eta\left(\theta - \frac{(n - 1)\pi}{N}\right) \qquad m \in \mathbb{N}_2 \quad n \in \mathbb{N}_{2N+1}$$

where we have replaced $2\tau + \nu + n$ by $n - 1$ and set $\mathbb{N}_d = \{0, 1, \ldots, d\}$.

5. Remarks and discussion

(1) In general, considering the Iwasawa decomposition KAN of S, with K being the maximal compact subgroup, we have $S_0 \subset K \subset S$. In the case where $S_0 = K$ and $\mathcal{X}_I$ is a discrete group of transformations of $\mathcal{O}^*$, there is an alternative way to construct the frame generator [13, 14].

Let us consider the canonical projections

$$\pi : S \to \mathcal{O}^* \qquad \pi_1 : S \to \mathcal{X}_I \backslash S \qquad \text{and} \qquad \pi_2 : \mathcal{O}^* \to \mathcal{X}_I \backslash \mathcal{O}^*$$

and form the compact set of double equivalence classes modulo $(\mathcal{X}_I, K)$

$$\mathcal{X}_I \backslash S/K \simeq C/K \simeq \mathcal{X}_I \backslash NA \simeq \mathcal{X}_I \backslash \mathcal{O}^*.$$

Then, via the canonical projection $\pi_2 : \mathcal{O}^* \to \mathcal{X}_I \backslash \mathcal{O}^*$, there is a compact subset $\mathcal{Q}'$ in $\mathcal{O}^*$ such that

$$\pi_2(\mathcal{Q}') = \mathcal{X}_I \backslash \mathcal{O}^* \qquad \text{and} \qquad \mathcal{O}^* = \bigcup_{\tau \in I} s_\tau[\mathcal{Q}'].$$

In this way, a second frame generator $(\mathcal{X}_I, \mathcal{Q}')$ is determined. In fact, it can be easily shown that the two frame generators are identical. Consider the commutative diagram

$$\begin{array}{ccc} S = \chi_I C & \xrightarrow{\quad \pi \quad} & \mathcal{O}^* = S/K \\ \pi_1 \big\downarrow & & \big\downarrow \pi_2 \\ \chi_I \backslash S & \xrightarrow{\quad \pi_3 \quad} & \chi_I \backslash \mathcal{O}^* = \chi_I \backslash H/K \end{array} .$$

It enables us to write

$$\pi_2(\mathcal{Q}) = \mathcal{X}_I[\mathcal{Q}] \subset \mathcal{X}_I[\mathcal{Q}'] = \pi_2(\mathcal{Q}')$$

and this implies that $\mathcal{Q} \subset \mathcal{Q}'$. The first thing we can deduce *a priori* is that the size (by cardinality) of the frame generated by $(\mathcal{X}_I, \mathcal{Q}')$ is greater than that of the frame generated by $(\mathcal{X}_I, \mathcal{Q})$. Now, the problem hinges upon the choice of representatives of equivalence classes. A deeper analysis makes this sharper. Let C_0 be a fundamental domain of $\mathcal{X}_I$ in S. Then we have

$$\begin{aligned} \pi_3 \circ \pi_1(C_0) &= \mathcal{X}_I \backslash \mathcal{O}^* = \mathcal{X}_I[\mathcal{Q}'] = \pi_2(\mathcal{Q}') \\ &= \pi_2 \circ \pi(C_0) = \pi_2(C_0 K) \\ &\subset \pi_2 \circ \pi(C) = \pi_2(CK) = \pi_2(\mathcal{Q}). \end{aligned}$$

With this we have $\mathcal{Q}' \subset \mathcal{Q}$. Thus, $\mathcal{Q} = \mathcal{Q}'$ and it is clear that the two frame generators $(\mathcal{X}_I, \mathcal{Q})$ and $(\mathcal{X}_I, \mathcal{Q}')$ are in fact identical.

(2) The results obtained in this paper open up new possibilities, by enlarging the class of the manifolds on which one could analyse signals. In addition, they show that the construction of Aniello *et al* remains applicable (with slight modifications) to general semidirect product groups with induced UIRs, which are square integrable modulo a subgroup. However, some questions remain open.

- A further extension is possible, since here we only consider the case where the group S is semisimple and connected. The only effect of this assumption was to provide a uniform lattice of G. In fact, there is no need for the discrete subset taken in G to have a group structure. As a confirmation, all the known constructions in the literature have been done with subsets without a group structure, or even a definite structure.

- We have not discussed here the density problem of the set $\{(q^j_{n,l}, p_n) : j = 1, \ldots, N; l \in \mathbb{Z}^m; n \in \Theta\}$ in Γ in connection with the Shannon sampling theorem.
- In view of the above, it would be interesting to check the efficiency of the frames for numerical computations. Even formally, it would be difficult to say anything about the commonly used efficiency test $|B/A - 1| \ll 1$, since the frame bounds found in (4.22) depend, to a certain extent, on the quality (precision) of the approximations performed. However, in the case where $K_j = K, \forall j$, the estimation of B/A gives the indication that good candidates for the choice of η should be band-limited functions with thin widths [15].
- Again, for implementation purposes, it would be useful to characterize completely the uniform lattice $\mathcal{X}_I$ and the quotient space $S/\mathcal{X}_I$. We have exhibited a concrete example in this paper for the Euclidean group $E(2)$, but for other groups, such as the Poincaré group $\mathcal{P}^{\uparrow}_{+}(1, 3)$, the exercise is less obvious [9]. For general semisimple, connected Lie groups, there are straightforward examples of discrete subgroups which belong to the centre of the group. But they are too small. Finding sufficiently large and nontrivial discrete subgroups of that kind of groups is a difficult problem.

  Note that the covering of S^1 in formula (4.23) is not optimal, in the sense that we did not care to control the extent of the overlaps of the shifted copies of J. As is well known, this is the main source of the redundancy contained in the reconstruction formulae.

Appendix. Proof of lemma 4.3

(i) The vectors $\{e^j_{n,l}\}$ are pairwise orthogonal and of norm 1.

$$
\begin{aligned}
\langle e^j_{n,l}, e^j_{n,l'}\rangle &= \int_{\mathcal{O}^*} \mathrm{d}\nu(k)\rho_n(k)\overline{e^j_{n,l}(k)}e^j_{n,l'}(k) \\
&= (\mu^j)^{-1}\int_{\Lambda(p_n)[K_j]} \mathrm{d}\nu(k)\rho_n(k)\exp\left[-\mathrm{i}\langle F(p_n)^*\sigma(p_n)^{-1}[k], q^j_{n,l'} - q^j_{n,l}\rangle\right] \\
&= (\mu^j)^{-1}\int_{s_0(p_n)[K_j]} \mathrm{d}\nu(k)\rho_n(\sigma(p_n)[k])\exp\left[-\mathrm{i}\langle F(p_n)^*k, q^j_{n,l'} - q^j_{n,l}\rangle\right] \\
&\qquad (k \mapsto \sigma(p_n)[k] + \text{measure invariance}) \\
&= (\mu^j)^{-1}\int_{F(p_n)^*s_0(p_n)[K_j]} \mathrm{d}\nu(k)\frac{m(F(p_n)^{*-1}k)}{m(k)}|\mathcal{I}(p_n, F(p_n)^{*-1}k)|^{-1} \\
&\qquad \times \rho_n(\sigma(p_n)[F(p_n)^{*-1}k])\exp\left[-i\langle k, q^j_{n,l'} - q^j_{n,l}\rangle\right] \\
&\qquad (\text{admissibility of } \widehat{\sigma} \text{ and } k \mapsto F(p_n)^{*-1}k) \\
&= (\mu^j)^{-1}\exp\left[-\mathrm{i}\langle k_0, q^j_{n,l'} - q^j_{n,l}\rangle\right] \\
&\qquad \times \int_{F(p_n)^*s_0(p_n)[K_j]} \mathrm{d}\nu(k)\frac{m(F(p_n)^{*-1}k)}{m(k)}|\mathcal{I}(p_n, F(p_n)^{*-1}k)|^{-1} \\
&\qquad \times \rho_n(\sigma(p_n)[F(p_n)^{*-1}k])\exp\left[-\mathrm{i}2\pi\sum_{\kappa=1}^{m}[\text{measure}(R_{j,\kappa})]^{-1}(l'_\kappa - l_\kappa)\psi(k)_\kappa\right] \\
&= (\mu^j)^{-1}\exp\left[-\mathrm{i}\langle k_0, q^j_{n,l'} - q^j_{n,l}\rangle\right]\int_{R_j}\mathrm{d}x\left|\frac{\mathrm{d}\psi(k)}{\mathrm{d}\nu(k)}\right|^{-1}\frac{m(F(p_n)^{*-1}k)}{m(k)} \\
&\qquad \times |\mathcal{I}(p_n, F(p_n)^{*-1}k)|^{-1}\rho_n(\sigma(p_n)[F(p_n)^{*-1}k])
\end{aligned}
$$

$$\times \exp\left[-\mathrm{i}2\pi \sum_{\kappa=1}^{m} [\mathrm{measure}(R_{j,\kappa})]^{-1}(l'_\kappa - l_\kappa)x_\kappa\right]$$

(by the change $k \mapsto \psi^{-1}(x)$)

$$= (\mu^j)^{-1} \exp\left[-\mathrm{i}\langle k_0, q^j_{n,l'} - q^j_{n,l}\rangle\right] \int_{R_j} \mathrm{d}x$$

$$\times \exp\left[-\mathrm{i}2\pi \sum_{\kappa=1}^{m} [\mathrm{measure}(R_{j,\kappa})]^{-1}(l'_\kappa - l_\kappa)x_\kappa\right]$$

$$\left(\text{because } \rho_n(\sigma(p_n)[F(p_n)^{*-1}k]) = |\mathcal{I}(p_n, F(p_n)^{*-1}k)| \left|\frac{\mathrm{d}\psi(F(p_n)^{*-1}k)}{\mathrm{d}\nu(F(p_n)^{*-1}k)}\right|\right)$$

$$= \exp\left[-\mathrm{i}\langle k_0, q^j_{n,l'} - q^j_{n,l}\rangle\right]\delta_{l,l'}.$$

(ii) The set is total.

For $\phi \in L^2(\Lambda(p_n)[K_j], \rho_n(k)\,\mathrm{d}\nu(k))$ and $l \in \mathbb{Z}^m$, using similar arguments as above, we have

$$\langle e^j_{n,l}, \phi\rangle = \int_{\mathcal{O}^*} \mathrm{d}\nu(k)\rho_n(k)\overline{e^j_{n,l}(k)}\phi(k)$$

$$= (\mu^j)^{-1/2} \int_{\Lambda(p_n)[K_j]} \mathrm{d}\nu(k)\rho_n(k) \exp[-\mathrm{i}\langle\sigma(p_n)^{-1}[k], \varphi(p_n)\rangle]$$

$$\times \exp\left[\mathrm{i}\langle F(p_n)^*\sigma(p_n)^{-1}[k], q^j_{n,l}\rangle\right]\phi(k)$$

$$= (\mu^j)^{-1/2} \int_{s_0(p_n)[K_j]} \mathrm{d}\nu(k)\rho_n(\sigma(p_n)[k]) \exp[-\mathrm{i}\langle k, \varphi(p_n)\rangle]$$

$$\times \exp\left[\mathrm{i}\langle F(p_n)^*k, q^j_{n,l}\rangle\right]\phi(\sigma(p_n)[k])$$

$$= (\mu^j)^{-1/2} \int_{F(p_n)^*s_0(p_n)[K_j]} \mathrm{d}\nu(k)\frac{m(F(p_n)^{*-1}k)}{m(k)}|\mathcal{I}(p_n, F(p_n)^{*-1}k)|^{-1}$$

$$\times\ \rho_n(\sigma(p_n)[F(p_n)^{*-1}k]) \exp[-\mathrm{i}\langle F(p_n)^{*-1}k, \varphi(p_n)\rangle]$$

$$\times \exp\left[\mathrm{i}\langle k, q^j_{n,l}\rangle\right]\phi(\sigma(p_n)[F(p_n)^{*-1}k])$$

$$= \exp\left[\mathrm{i}\langle k_0, q^j_{n,l}\rangle\right](\mu^j)^{-1/2} \int_{F(p_n)^*s_0(p_n)[K_j]} \mathrm{d}\nu(k)\frac{m(F(p_n)^{*-1}k)}{m(k)}$$

$$\times\ |\mathcal{I}(p_n, F(p_n)^{*-1}k)|^{-1}\rho_n(\sigma(p_n)[F(p_n)^{*-1}k]) \exp\left[-\mathrm{i}\langle n^*_{0,F(p_n)^{*-1}k}, \varphi(p_n)\rangle\right]$$

$$\times \exp\left[\mathrm{i}2\pi \sum_{\kappa=1}^{m} [\mathrm{measure}(R_{j,\kappa})]^{-1} l_\kappa \psi(k)_\kappa\right]\phi(\sigma(p_n)[F(p_n)^{*-1}k])$$

$$\left(\text{because } V^* = V_0^* \oplus N_0^* \ni F(p_n)^{*-1}k = v^*_{0,F(p_n)^{*-1}k} + n^*_{0,F(p_n)^{*-1}k}\right)$$

$$= \exp\left[\mathrm{i}\langle k_0, q^j_{n,l}\rangle\right](\mu^j)^{-1/2} \int_{R_j} \mathrm{d}x \exp\left[-\mathrm{i}\langle n^*_{0,F(p_n)^{*-1}\psi^{-1}(x)}, \varphi(p_n)\rangle\right]$$

$$\times \exp\left[\mathrm{i}2\pi \sum_{\kappa=1}^{m} [\mathrm{measure}(R_{j,\kappa})]^{-1} l_\kappa x_\kappa\right]\phi(\sigma(p_n)[F(p_n)^{*-1}\psi^{-1}(x)]).$$

Thus,

$$\left\langle e^{j}_{n,l}, \phi\right\rangle = 0 \qquad \forall l \in \mathbb{Z}^m$$
$$\text{iff} \quad \exp\left[-\mathrm{i}\left\langle n^{*}_{0,F(p_n)^{*-1}\psi^{-1}(x)}, \varphi(p_n)\right\rangle\right]\phi(\sigma(p_n)[F(p_n)^{*-1}\psi^{-1}(x)]) = 0 \quad \text{for a.e. } x \in R_j$$
$$\text{iff} \quad \phi(\sigma(p_n)[F(p_n)^{*-1}\psi^{-1}(x)]) = 0 \qquad \text{for a.e.} \quad x \in R_j$$
$$\text{iff} \quad \phi(k) = 0 \qquad \text{for a.e.} \quad k \in \Lambda(p_n)[K_j].$$

□

References

[1] Aniello P, Cassinelli G, De Vito E and Levrero A 1998 Wavelet transforms and discrete frames associated to semi-direct products *J. Math. Phys.* **39** 3965–73

[2] Ali S T, Antoine J-P and Gazeau J-P 2000 *Coherent States, Wavelets and Their Generalizations* (New York: Springer)

[3] Aniello P, Cassinelli G, De Vito E and Levrero A 2001 On discrete frames associated with semi-direct products *J. Fourier Anal. Appl.* **7** 199–206

[4] Antoine J-P and Hohouéto A L 2003 Discrete frames of Poincaré coherent states in 1+3 dimensions *J. Fourier Anal. Appl.* **9** 141–72

[5] Raghunathan M S 1982 *Discrete Subgroups of Lie Groups* (Berlin: Springer)

[6] Bernier D and Taylor K F 1996 Wavelets from square-integrable representations *SIAM J. Math. Anal.* **27** 594–608

[7] Feichtinger H G and Gröchenig K H 1989 Banach spaces related to integrable group representations and their atomic decompositions *J. Funct. Anal.* **86** 307–40

[8] Bohnké G 1991 Treillis d'ondelettes associés aux groupes de Lorentz *Ann. Inst. Henri Poincaré* **54** 245–59

[9] Hohouéto A L 1999 On some discrete relativistic frames *PhD Thesis* Univ. Nat. du Bénin, Abomey-Calavi

[10] Gilmore R 1974 On properties of coherent states *Rev. Mex. Fis.* **23** 143–87

[11] Perelomov A M 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)

[12] De Bièvre S 1989 Coherent states over symplectic homogeneous spaces *J. Math. Phys.* **30** 1401–7

[13] Onishchick A L 1993 *Lie Groups and Lie Algebras I* (Berlin: Springer)

[14] Onishchick A L 2000 *Lie Groups and Lie Algebras II* (Berlin: Springer)

[15] Papoulis A 1977 *Signal Analysis* (New York: McGraw-Hill)